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A Time–Frequency Analysis Method for Radar Scattering

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Abstract—A time–frequency analysis method to study electromagnetic scattering is presented and demonstrated using canonical objects. The time–frequency analysis method utilizes the Bargmann transform to formulate the signal representation in phase space. The use of the Bargmann transform leads to an attractive parametric signal representation in terms of complex polynomials, and elliptical filters can be constructed to crop or extract selected areas of the phase plane. The signal representation and filtering operations are demonstrated using scattering responses from spheres and thin wires, and the prominent scattering features are identified and extracted.

Index Terms—Scattering, time–frequency analysis.

I. BACKGROUND

Electromagnetic signals are traditionally expressed in either the time domain or the frequency domain. However, with the development of quantum optics, a different formulation has been introduced to take into account the particle-like nature of quantized electromagnetic fields (e.g., photons) [1], [4]. The mathematical formalism is known as coherent-state analysis. The basic components of the analysis are the coherent states which are of the form

$$g^{(p, q)}(x) = \frac{1}{\pi^{1/4}} e^{ipx} e^{-ipq/2} e^{-(x-q)^2/2} \qquad (1)$$

where, in quantum mechanical terms, these are the photons that are characterized by a momentum p and a position coordinate q. Any signal $f(x)$ can be represented in the phase plane as $F(p, q)$ [4], [5] through the projections given by the well-known transform

$$F(p, q) = \int_{-\infty}^{\infty} f(x) \overline{g^{(p, q)}(x)} \, dx \qquad (2)$$

where the bar denotes the complex conjugate. The corresponding inverse transform [3], [4] is given by

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(p, q) g^{(p, q)}(x) \, dp \, dq. \qquad (3)$$

If one makes the identification of x with ct (i.e., $x \rightarrow ct$ where c is the velocity of light), then the above transform becomes the well-known windowed Fourier transform. In signal processing, the windowed transform provides a localized time–frequency picture of a signal $f(t)$. The momentum p corresponds to the angular frequency ω (i.e., $p \rightarrow \omega$), and the coordinate q corresponds to the center of the window transform τ (i.e., $q \rightarrow \tau$). In this paper, we elect to keep the physics-based identity of the analysis and blend it with the signal-

processing applications to foster a broader picture of the physical phenomena involved.

An attractive compact form of the coherent-state transform given by (2) can be developed with the aid of the Bargmann transform. In the Bargmann transform, the phase space coordinates q and p are combined together to form a complex variable $z = q - ip$. Utilizing the Bargmann transform, the phase space representation $F(p, q)$ of a function $f(x)$ becomes a complex function $F(z)$, and all the powerful techniques of complex variable theory are available for the representation and analysis of the transforms.

The Bargmann transform has been introduced by a number of investigators [1], [4]–[7], and is defined as

$$FBf(z) = \frac{1}{\pi^{1/4}} e^{-z^2/4} \int_{-\infty}^{\infty} e^{zx - x^2/2} f(x) \, dx. \qquad (4)$$

The Bargmann transform is an isometry from the space of square integrable functions $L^2(\mathbb{R})$ to the space $L^2(\mathbb{C}, e^{-|z|^2/2} dz)$ which is known as the Fock space $\mathcal{F}$. The Fock space is defined as

$$\mathcal{F} = \left\{ F: F \text{ is an entire function on } \mathbb{C}, \right.$$
$$\left. \|F\|_{\mathcal{F}}^2 = \int_{-\infty}^{\infty} |F(z)|^2 e^{-|z|/2} \, dz < \infty \right\} \qquad (5)$$

where $z = q - ip$, and q and p are the phase plane coordinates.

The Fock space is the space of entire analytic functions defined in the whole complex plane $\mathbb{C}$. Describing a dynamical system in terms of its space position coordinate q and momentum p is known as the phase space representation.

The Bargmann transform can also be written in a traditional form using the phase space variables p and q as

$$Bf(z) = \frac{1}{\pi^{1/4}} e^{(p^2+q^2)/4} e^{ipq/2} \int_{-\infty}^{\infty} e^{-ipx-(x-q)^2/2} f(x) \, dx. \qquad (6)$$

The corresponding inverse Bargmann transform in this form [1], [4] is

$$f(x) = B^{-1} e^{|z|^2/4} F(x)$$
$$= \frac{e^{-x^2/2}}{\pi^{1/4}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\overline{z}x} e^{-\overline{z}^2/4} F(z) [e^{-|z|^2/4} \, dz],$$
$$dz = dp \, dq. \qquad (7)$$

II. HERMITE-FUNCTION EXPANSIONS

The set of Hermite functions is the natural basis for the Bargmann transform. This represents an important analytical tool for the representation of the signals. The orthogonal Hermite functions $\varphi_n(x)$ are

$$\varphi_n(x) = \frac{e^{-\frac{x^2}{2}}}{\pi^{1/4}} \frac{H_n(x)}{\sqrt{2^n n!}} \text{ with the Bargmann transform}$$

$$\zeta_n(z) = \frac{z^n}{\sqrt{2^n n!}} \qquad (8)$$

where $H_n(x)$ is the Hermite polynomial. It is important to note that the Bargmann transform of the Hermite function is a monomial. Accordingly, an arbitrary function $f(x)$ represented in a Hermite function basis is

$$f(x) = \sum \alpha_n \varphi_n(x) \quad \alpha_n = \int_{-\infty}^{\infty} f(x) \varphi_n(x) \, dx \qquad (9)$$

with a phase space transform of the form

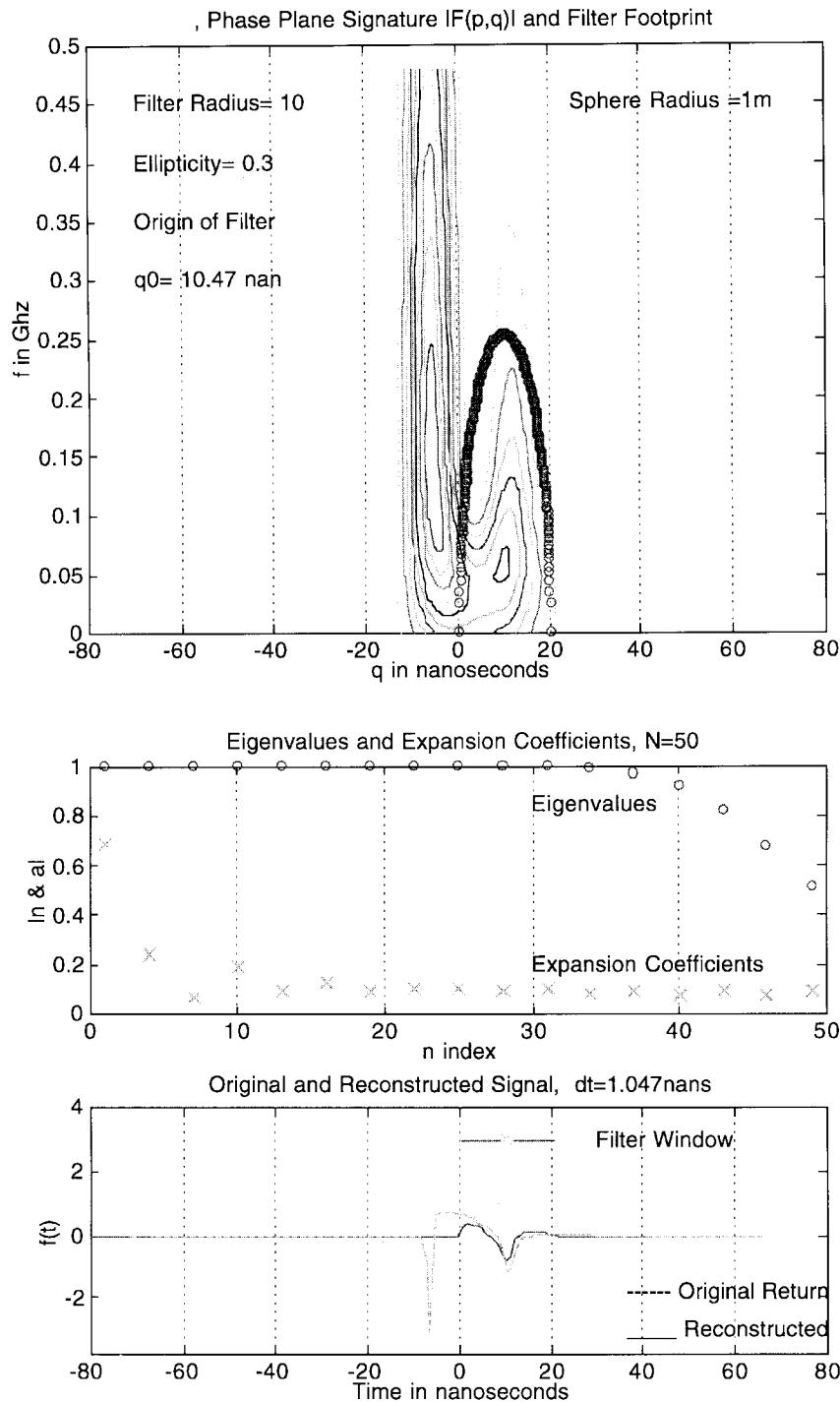$$F(z) = \sum \alpha_n \zeta_n(z). \qquad (10)$$

Fig. 1.   Electromagnetic impulse response of a perfectly conducting sphere. Phase space analysis and selective filtering of the creeping wave.

Notice that the phase space transform is a sum of weighted monomials or, in fact, a power series in z. This is in accordance with the basic premise of Bargmann who stated that the Bargmann transform projections are entire functions.

The expansion of signals in terms of the orthogonal Hermite functions enables one to carry out very efficient filtering in the phase plane. The filtering operator is defined in the inverse transform where the inverse integral in (7)

$$\mathbf{P}_{S_R} f(x) = \frac{e^{-x^2/2}}{\pi^{1/4}} \int \int_{S_R} e^{\bar{z}x} e^{-\bar{z}^2/4} F(z) [e^{-|z|^2/4} \, dz],$$

$$dz = dpdq \quad (11)$$

where S_R is a selected area in the phase plane. It has been shown [1], [4], [5] that the filtering operator for a selected finite area S_R is a traceless operator. This implies that a set of eigenfunctions and eigenvalues exist such that

$$\mathbf{P}_{S_R} \varphi_n(x) = \lambda_n \varphi_n(x). \quad (12)$$

However, for the special case where the filtering region is an elliptical disk, the eigenfunctions are the Hermite functions. Consequently, filtering can be implemented by finding an expansion of the signal in the x domain (or time domain) in terms of the Hermite functions, and weighing the coefficients with the eigenvalues of the filter operator
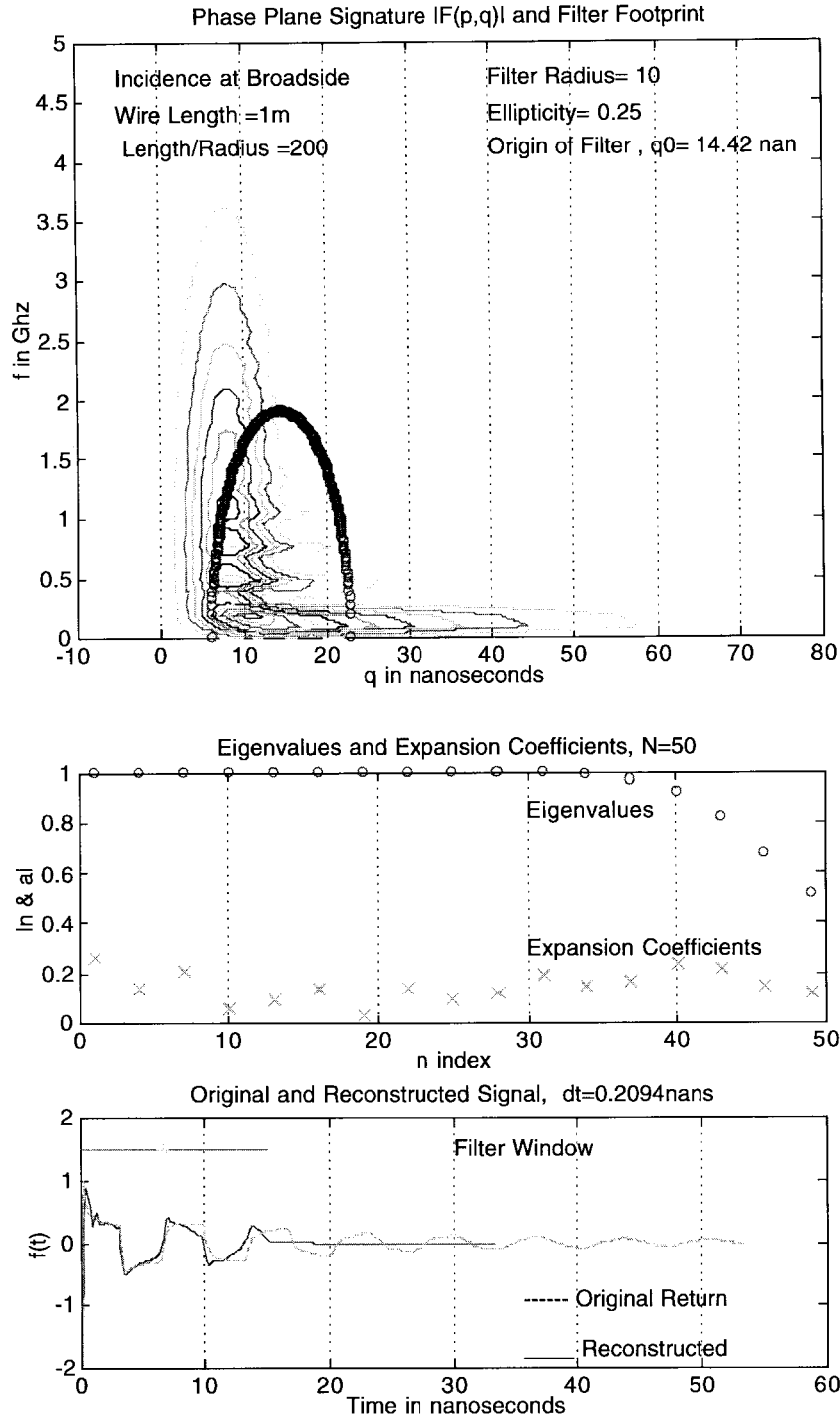
Fig. 2. Electromagnetic impulse response of a thin wire. Phase space analysis and selective filtering.

$\mathbf{P}_{Sk}$ as follows [5]:

$$\mathbf{P}_{S_R} f(x) = \sum_{n=0}^{R^2/2} \lambda_n \langle \varphi_n(x), \, f(x) \rangle \varphi_n(x) \qquad (13)$$

where R is the radius of the filter disk in the phase plane, λ_n is the eigenvalue

$$\lambda_n = \frac{1}{n!} \gamma(n+1, \, R^2/2) \qquad (14)$$

and γ is the incomplete gamma function.

The upper limit of the summation $N = \frac{R^2}{2}$ is proportional to the area of the filter. It has been shown [5] that, in general, the number of terms necessary to capture the filtered signal are

$$N = \frac{\text{Filter area}}{2\pi}. \qquad (15)$$

The number of terms N is also known as the degrees of freedom of the filtered signal.

The Hermite expansions are very robust and easy to implement numerically. The corresponding monomials are also orthogonal in the phase plane and provide a convenient basis for expanding the signals.

Furthermore, the monomials ideally exhibit the analytical properties of the transformed signals.

III. APPLICATIONS

Two examples have been chosen to demonstrate the use of the phase space signal representation and filtering. In both examples, the signal was the calculated theoretical electromagnetic impulse response. The impulse response was approximated by taking the inverse Fourier transform of an extremely broad-band frequency-domain representation of the normalized scattering amplitude of the electric field (see Appendix A) due to an incident plane wave. The first target was the sphere for which the exact solution is known in terms of the Mie series [2, p. 397]. The second example is the thin wire where we used the Ufimtsev approximate solution [2, p. 483]. These two examples represent two extremes. The scattered signal from the sphere is mainly due to a specular return and creeping-wave body scattering, whereas the scattered signal from the thin wire is due mainly to resonance.

An example of filtering the scattering response of a perfectly conducting sphere of radius $a = 1m$ is shown in Fig. 1. The impulse response of the sphere has a sharp specular peak followed by the creeping wave which appears at a time $\tau = (2a + \pi)/c$. An elliptic filter has been chosen in phase space with $R = 10$ and ellipticity $\alpha = 0.3$ to capture most of the energy due to the creeping wave. The filter window and the filtered signal are shown in Fig. 1. The filtered signal closely approximates the portion of the signal desired for extraction.

Scattering from the wire of length $L = 1m$ is shown in Fig. 2. The return of the wire is characterized by a long ringing trail with the period of the oscillation given by $2L/c$. A filter of radius $R = 10$ and ellipticity $\alpha = 0.3$ was placed to capture the first few cycles of the ringing. The filtered response illustrates that the main features of the selected region are well-approximated.

IV. CONCLUSIONS

The coherent-state analysis in combination with the Bargmann transform allows the representation of signals in the phase plane where both the time and frequency features can be displayed simultaneously through a mathematically compact description. Distinguishing features can be extracted by placing elliptical filters in the selected areas of the phase plane. For the special case of elliptical filters, the filtering can take place in the time domain by using the proper Hermite function expansions. The ellipticity and the placement of the filters can be chosen as to accentuate the temporal features of targets like the sphere or the resonant frequency features of targets like that of the ringing wire. The placement of the filters is arbitrary and can be positioned anywhere in the phase plane. The advantage of the suggested method is that it can be performed in the time domain without having to transform into the phase plane, crop or localize, and then transform back into the time domain.

APPENDIX A

A. Scattering from a Sphere

Bowman [2, p. 397] has shown that the scattered electric field on axis from a plane wave incident on a perfectly conducting sphere is

$$E^S(k) = \frac{e^{ikr}}{r} \left[\frac{S(k)}{k} \right] \qquad (A.1)$$

where k is the wavenumber and r is the range. The quantity of interest here is the scattering amplitude $f(k) = \left[\frac{S(k)}{k} \right]$, which is given by the Mie series

$$\left[\frac{S(k)}{k} \right] = -\frac{i}{k} \sum_{n=1}^{\infty} (-1)^n (n + 1/2)(b_n - a_n) \qquad (A.2)$$

where

$$a_n = \frac{j_n(ka)}{h_n^{(2)}(ka)} \quad b_n = \frac{\frac{d}{dka}[kaj_n(ka)]}{\frac{d}{dka}[kah_n^{(2)}(ka)]} \quad k = \omega/c \qquad (A.3)$$

and c is the velocity of light. Typically, $2|ka|$ terms are needed for the sum to converge. The inverse Fourier transform of the scattering amplitude approximates the time-domain impulse response for the sphere.

B. Scattering from a Thin Wire

Umifetz [2, p. 397] has shown that the scattering amplitude for a thin wire due to a plane wave at broadside incidence is

$$
\begin{aligned}
\left[\frac{S(k)}{k} \right] &= \frac{1}{k} \frac{2i}{(\Omega_0 + 2\ln 2)^2} \{ -1 + (\Omega_0 + \ln 2)(1 + iL) \\
&\quad - 2e^{2iL}(\Omega_0 + 2\ln 2)^2 g(2L, \pi/2) \} \\
&\quad \times \left[1 - g(2L, \pi/2) \left\{ 1/2 + iLT(2L) - \frac{e^{2iL}\Omega_0 g(2L, \pi/2)}{1 + e^{2iL}\Omega_0 g(2L, 0)} \right\} \right]
\end{aligned}
$$
$$(A.4)$$

where

$$\Omega_0 = -2\ln ka - 2\gamma + i\pi \qquad (A.5)$$

a is the radius of the wire, k is the wavenumber, $\gamma = 0.57731566$,

$$g(x, \theta) = [\Omega_0 + l(2x) + T(2x \cos^2 \theta/2)] \qquad (A.6)$$

$$l(x) = \gamma + \ln x - i\pi/2$$

$$T(x) = -e^{-ix}[Ci(x) + iSi(x) - i\pi/2]$$

$$Ci(x) = \int_x^{\infty} \frac{\cos \xi}{\xi} d\xi$$

$$Si(x) = \int_0^x \frac{\sin \xi}{\xi} d\xi. \qquad (A.7)$$

As in the procedure for the sphere, the inverse Fourier transform of the scattering amplitude was used to approximate the time-domain impulse response for the wire.

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